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Feynman Path Integral of the Chern-Simons Action and Invariants for Three-Manifolds

by

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Introduction

In recent years, gauge theory and quantum field theory have provided various fruitful ideas and tools for the study of invariants for low-dimensional manifolds (see [2], [3]). Indeed, we have some brilliant theories; the Donaldson theory in 4 dimensions, [6], the Floer-Taubes theory on Casson invariant for homology 3-spheres, [8], [13], and Witten's theory on the Jones polynomial, [15].

In this paper, we would like to find some invariants for three-manifolds by using an idea in quantum field theory. Let M be a compact, oriented three-manifold and let P be the trivial principal $SU(2)$-bundle on M. We denote by $\mathscr{C}$ the space of all gauge equivalence classes of connections on P and by L the Chern-Simons action defined on the affine space of all connections A on P (see §1 below). Then the 'partition function' is defined by the Feynman path integral

$$Z(M; k)=\int_{\mathscr{C}} \exp(ikL(A))\mathscr{D}[A] ,$$

where k is a positive integer and $\mathscr{D}[A]$ denotes 'Feynman measure' on $\mathscr{C}$. One often says that since the Chern-Simons action is independent of any metric on M the Feynman path integral gives rise to a topological invariant. However, there is a serious question in this statement. In fact, there is no mathematically rigorous definition of 'Feynman measure'. Hence we have to ask whether the Feynman path integral makes sense.

In [15], Witten tries to evaluate the large k limit of the integral under a certain strong condition (see also [4] Chap. 7). To reduce the problem to Gaussian integral, gauge fixing is needed. So he takes a Riemannian metric on M and, by using the results of Atiyah *et al.*, [5], [11], he arrives at a plausible result (see §1). Unfortunately, his result depends on the choice of a metric. As is pointed out by Witten himself, there is an important gap in his reasoning. The fundamental manifold S^3 does not satisfy his cohomological condition. So he proposes to extract the large k contribution of arbitrary flat connections without any cohomological assumption.

Taking account of this problem of Witten and his result, we shall interpret and evaluate the Feynman path integral so that $Z(M; k)$ gives a topological invariant. After long deliberation, we have decided to study the problem without using any metric on M. This standpoint fits perfectly to the main purport of the topological quantum field theory of Witten. However, it yields some difficulties. The main difficulty arises in gauge fixing.

Although the crucial point of the problem lies in the concept 'Feynman measure', we do not intend to clarify the true content of the concept. However, it is certain that our cumbersome main task is to define a peculiar measure space for evaluating the integral. Our study is then based on the following assumptions and requirements: a) 'Feynman measure' is an ordinary measure and is independent of any metric structure on M. b) 'Feynman measure' on $(0, \infty)$ is either the Gaussian type $\exp(-x^2)dx$ or the Laplace type $\exp(-x)dx$. c) Since the Chern-Simons action is fairly simple, the Feynman path integral should be carried out in a concrete form. d) The idea should be simple and natural. As for an infinite-dimensional vector space, we shall often replace it by a specious finite-dimensional vector space.

In order to obtain a reasonable result, we have to impose a certain finiteness condition on the moduli space $\mathscr{M}$ of flat connections on P. Then our final result is roughly presented as follows:

$$Z(M; k) = (k^2+1)^{-\lambda(M)/2} \sum_j \exp(ikL(A_j)) \cdot \chi_j(k),$$

where $\lambda(M)$ is a half-integer invariant of M, A_j's are suitable flat connections on P and $\chi_j(k)$'s are complex numbers of the form

$$\chi_j(k) = \frac{-1}{\sqrt{2}} (\Lambda_j^+(k)\exp(i\lambda(M)\theta(k)) + \Lambda_j^-(k)\exp(-i\lambda(M)\theta(k))).$$

We can see that the quantity $Z(M; k)$ thus obtained is a topological invariant depending on the orientation of M. In fact, let M^* denote the manifold M with the opposite orientation. Then $Z(M^*; k)$ is equal to the conjugate complex number of $Z(M; k)$ (cf. [3] p. 181). It should be also remarked that if k is sufficiently large then $Z(S^3; k) \sim k^{-3/2}$. This is, indeed, a part of Witten's conjecture (see [15] p. 362).

The paper is divided into five sections. Our problem is restated in a precise form in Section 1. Section 2 is occupied with the study of gauge invariants for flat connections. Then a measure space on $\mathscr{M}$ and 'Feynman measure' of some sets are introduced in Section 3. Section 4 is devoted to evaluating the integral. Some elementary properties of main quantities are shown in the last section.

I would like to express my gratitude to Mr. Ano, the young theoretical physicist, who has taught me quantum field theory and has helped me with many interesting discussions.

§1. Feynman path integral of the Chern-Simons action

To state our problem in a precise form, we have to recall the rudiments of gauge theory in three dimensions (cf. [8], [13]).

Throughout this paper, we denote by M a compact, connected, oriented three-manifold and by $\mathscr{A}$ the affine space of all connections on the trivial principal $SU(2)$-bundle P on M. Let su_2 denote the Lie algebra of $SU(2)$. We can identify $\mathscr{A}$ with the vector space $\Omega^1(su_2)$ of all su_2-valued 1-forms on M in such a way that the trivial connection on P corresponds to the zero form in $\Omega^1(su_2)$. Then the curvature of a connection A is the su_2-valued 2-form on M given by $F(A)=dA+A\wedge A$. A is said to be flat if $F(A)=0$. The gauge transformation group $\mathscr{G}$ of P is naturally identified with the group of all smooth maps g from M to $SU(2)$. $\mathscr{G}$ acts on $\mathscr{A}$ by the non-linear transformation law

$$A\cdot g=g^{-1}dg+g^{-1}Ag\,, \qquad g\in\mathscr{G}\,.$$

Then we get $F(A\cdot g)=g^{-1}F(A)g$. Hence $\mathscr{G}$ acts also on the space $\mathscr{F}$ of all flat connections on P. By the action, we can consider the orbit spaces $\mathscr{C}=\mathscr{A}/\mathscr{G}$ and $\mathscr{M}=\mathscr{F}/\mathscr{G}$. It turns out that $\mathscr{C}$ is an infinite-dimensional manifold except for some singular points associated to reducible connections (see [13]). $\mathscr{M}$ is called the moduli space of flat connections. We are much interested in $\mathscr{C}$ and $\mathscr{M}$ rather than $\mathscr{A}$ and $\mathscr{F}$ because the group $\mathscr{G}$ is too large to study properly. So every system on $\mathscr{A}$ or $\mathscr{F}$ should be gauge invariant. Needless to say, this is the fundamental principle of the theory. However, as $\mathscr{A}$ is an affine space, it may be convenient to study $\mathscr{C}$ as a submanifold of $\mathscr{A}$. This idea leads us to the problem of gauge fixing (cf. [9]).

In [14] and [15], E. Witten has proposed a topological quantum field theory, in which he asserts that all observables of a quantum field theory should be topological invariants of a background manifold. In other words, the theory requires that any quantity should be computed without any *a priori* choice of a metric and any other geometric structure on the manifold (see [3] and also [4]). To develop such a quantum field theory, we have to choose a 'topological action'. In 3 dimensions, there is a natural action; namely the Chern-Simons action:

$$L(A)=\frac{1}{4\pi}\int_M \mathrm{Tr}(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)\,,$$

where Tr denotes the trace of 2×2 complex matrix. The differential $(dL)_A$ of L at $A\in\mathscr{A}$ is given by

$$(dL)_A(B)=\frac{1}{2\pi}\int_M \mathrm{Tr}(B\wedge F(A))\,, \qquad B\in\Omega^1(su_2)\,.$$

Here we regard $\Omega^1(su_2)$ as the tangent vector space $T_A(\mathscr{A})$ to $\mathscr{A}$ at A. It follows that A is a critical point of L if and only if A is flat. So $\mathscr{F}$ is nothing but the set of all critical points of L. This simple fact will play an important role in our study. By the Bianchi identity, $(dL)_A$ annihilates the tangent vector space to the $\mathscr{G}$-orbit through A. Thus we can think of dL as being the pull back of a 1-form on $\mathscr{C}$. But L itself is not gauge invariant. In fact, for each $g\in\mathscr{G}$, $L(A)$ is transformed as follows:

$$L(A\cdot g)=L(A)-2\pi\cdot\deg(g)\,, \tag{1.1}$$

where $\deg(g)$ denotes the mapping degree of $g\colon M\to SU(2)$. Although $L(A)$ does not descend to $\mathscr{C}$ as a function, $L(A)$ appears in quantum field theory in the form $\exp(ikL(A))$, where k is a real number corresponding to the reciprocal of Planck's constant. From (1.1), we see that $\exp(ikL(A))$ is gauge invariant if and only if k is an integer. The parameter k is thus quantized. So, henceforth, we assume that k is a positive integer. Our task is then to interpret and evaluate the Feynman path integral

$$Z(M;k)=\int_{\mathscr{C}}\exp(ikL(A))\mathscr{D}[A]\,, \tag{1.2}$$

where $\mathscr{D}[A]$ denotes 'Feynman measure' or 'Feynman volume form' on $\mathscr{C}$.

Here we make a notational remark. Let X be a set. Given an equivalence relation in X and a point x of X, we always denote by $[x]$ the equivalence class determined by x.

Under some strong conditions, Witten has evaluated in [15] the large k limit of the integral (1.2) by using the idea of A. Schwarz, [12]. The crucial point lies in the use of Gaussian integral on a vector space $\mathscr{V}$:

$$\int_{\mathscr{V}}\exp(iQ(x))\mathscr{D}x=|\det(Q)|^{-1/2}\exp(i\pi\,\mathrm{sgn}(Q)/4)\,,$$

where $Q(x)$ is a non-degenerate quadratic form on $\mathscr{V}$. To reduce the problem to Gaussian integral, Witten takes a Riemannian metric on M and defines a certain self-adjoint operator K (kinetic operator) acting on an infinite-dimensional space of forms. To interpret $\det(K)$ and $\mathrm{sgn}(K)$, he uses the result of Ray and Singer, [11], and the η-invariants of Atiyah, Patodi and Singer, [5]. Let $(A_1,\cdots,A_n)$ be a complete system of gauge inequivalent flat connections. Then his result is

$$Z(M;k)=\exp(i\pi\eta(0)/2)\sum_j\exp(i(k+2)L(A_j))\cdot T(A_j)\,,$$

where $\eta(0)$ is the η-invariant of the trivial connection and $T(A_j)$ is the Ray-Singer analytic torsion (see [4]). Unfortunately, his result depends on the choice of a metric on M.

We shall study the Feynman path integral (1.2) under the following key assumption: 'Feynman measure' is an ordinary measure and is independent of any geometric structure on M. Then we can show formally that $Z(M;k)$ is a topological invariant depending on the orientation of M. Our main problem is therefore to evaluate the integral (1.2) so that we obtain a precise definition of such an invariant. We try to study the problem without using any metric on M. A metric will be used only in a subsidiary consideration. Although this standpoint fits to the topological quantum field theory, it yields some difficulties. Of course, the real difficulty lies in the lack of good understanding of infinite-dimensional spaces. In our study, however, infinite-dimensional vector spaces under consideration will appear in the form $\mathscr{V}/\mathscr{W}$, where $\mathscr{V}$ is an infinite-dimensional vector space and $\mathscr{W}$ is an infinite-dimensional linear subspace of $\mathscr{V}$. Then, to each vector space $\mathscr{V}/\mathscr{W}$, there corresponds a natural finite-dimensional vector space $H=\mathscr{W}/\mathscr{K}$. We shall use effectively H instead of $\mathscr{V}/\mathscr{W}$.

§2. Gauge invariants for flat connections

To study gauge invariants, we first construct a 'twisted de Rham complex'. Let V be a finite-dimensional real vector space. Consider a representation ρ of $SU(2)$ in V, $\rho: SU(2)\to GL(V)$, and its differential representation $\rho: su_2\to gl(V)$, where $gl(V)$ is the Lie algebra of all endomorphisms of V. Let $\Omega^p(V)$ denote the real vector space of all V-valued p-forms on M, $0\leqq p\leqq 3$. For a connection A on P, let $d_A^{(p)}: \Omega^p(V)\to \Omega^{p+1}(V)$ denote the covariant exterior differential operator defined by

$$d_A^{(p)}\zeta = d\zeta + \rho(A)\wedge\zeta\,, \qquad \zeta\in\Omega^p(V)\,.$$

We also denote it simply by d_A. For any $\zeta\in\Omega^p(V)$, we first have

$$dd_A\zeta + d_A d\zeta = \rho(dA)\wedge\zeta\,. \tag{2.1}$$

It follows from the Ricci identity that if A is flat then the sequence

$$\Omega^0(V) \xrightarrow{d_A} \Omega^1(V) \xrightarrow{d_A} \Omega^2(V) \xrightarrow{d_A} \Omega^3(V) \tag{2.2}$$

is a cochain complex. Notice that $\Omega^p(V)$ is identified with the vector space of all smooth sections of the vector bundle $\bigwedge^p T^*M\otimes V$ on M.Then it is a routine matter to show that (2.2) is an elliptic complex. Hence the p-th cohomology group $H_A^p(M, V)$ of (2.2) is finite-dimensional. On the other hand, for each $g\in\mathscr{G}$, let $\rho(g)$ denote the automorphism of $\Omega^p(V)$ determined by the composition $\rho\cdot g$. Then it is not hard to verify

$$d_A\cdot\rho(g) = \rho(g)\cdot d_{Ag}\,.$$

This implies that the map $\rho(g)$ is a cochain isomorphism from $(\Omega^*(V), d_{Ag})$ to $(\Omega^*(V), d_A)$. Hence,

$$\dim H_{Ag}^p(M, V) = \dim H_A^p(M, V)\,. \tag{2.3}$$

Summarizing these facts, we have

LEMMA 2.1. *Let A be a flat connection on P. Then the sequence* (2.2) *is a cochain complex, and the p-th cohomology group $H_A^p(M, V)$ is finite-dimensional. Moreover, the dimension of $H_A^p(M, V)$ satisfies* (2.3) *for any $g\in\mathscr{G}$.*

For the trivial connection $A=0$, we write simply $H^p(M, V)=H_0^p(M, V)$. By virtue of this lemma, we can find various kinds of integer-valued, or quantized, gauge invariants for flat connections.

First of all, consider the adjoint representation of $SU(2)$. Let D_A denote the differential operator associated with the representation. Then, for any $B \in \Omega^p(su_2)$, $D_A B$ is written as

$$D_A B = dB + A \wedge B - (-1)^p B \wedge A . \tag{2.4}$$

From Lemma 2.1, we have immediately the following

PROPOSITION 2.1. *Let A be a flat connection on P. Then $(\Omega^*(su_2), D_A)$ is a cochain complex, and the p-th cohomology group $H_A^p(M, su_2)$ is finite-dimensional. Moreover, the dimension $d^p(A)$ of $H_A^p(M, su_2)$ is a gauge invariant for flat connections.*

Let A be a flat connection on P. Then $H_A^1(M, su_2)$ may be considered as the (formal) tangent vector space to $\mathcal{M}$ at $[A]$, so $d^1(A)$ represents the local dimension of $\mathcal{M}$ at $[A]$.

In order to introduce another quantity which measures 'local size' of $\mathcal{M}$, we consider another natural representation of $SU(2)$. Recall that the quaternion field $\boldsymbol{H}$ plays an important role in the instanton construction (see [1]). To represent $\boldsymbol{H}$ as a subalgebra of the algebra gl_2 of all 2×2 complex matrices, we use Pauli's spin matrices:

$$\sigma_1 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, let σ_0 denote the identity matrix of degree 2. Then, as is well-known, $\boldsymbol{H}$ is identified with the subalgebra of gl_2 consisting of all elements of the form

$$X = x_0 \cdot \sigma_0 + x_1 \cdot \sigma_1 + x_2 \cdot \sigma_2 + x_3 \cdot \sigma_3 ,$$

where x_j's are arbitrary real numbers. It should be remarked that $SU(2)$ is identified with the group $Sp(1)$ of all quaternions with unit norm and that $(\sigma_1, \sigma_2, \sigma_3)$ forms a specific basis of the Lie algebra su_2.

Now, let $\rho : SU(2) \to GL(\boldsymbol{H})$ denote the representation given by

$$\rho(g)X = g \cdot X , \qquad g \in SU(2) , \quad X \in \boldsymbol{H} ,$$

and let δ_A denote the operator associated with the representation. Then, for any $B \in \Omega^p(\boldsymbol{H})$, δ_A is written as

$$\delta_A B = dB + A \wedge B . \tag{2.5}$$

Thus $\delta_A B$ consists of the first two terms of $D_A B$. From Lemma 2.1, we have immediately the following

PROPOSITION 2.2. *Let A be a flat connection on P. Then $(\Omega^*(\boldsymbol{H}), \delta_A)$ is a cochain complex, and the p-th cohomology group $H_A^p(M, \boldsymbol{H})$ is finite-dimensional. Moreover, the dimension $h^p(A)$ of $H_A^p(M, \boldsymbol{H})$ is a gauge invariant for flat connections.*

As is easily seen from (2.4) and (2.5), the operators D_A and δ_A can be defined on $\Omega^p(gl_2)$. Then, for any $B \in \Omega^p(gl_2)$ and any $C \in \Omega^q(gl_2)$, we find

$$\delta_A(B\wedge C)=(D_AB)\wedge C+(-1)^pB\wedge\delta_AC\,.$$

Before going further, we introduce a bilinear form on $\Omega^*(gl_2)$. Let $B\in\Omega^p(gl_2)$ and $B'\in\Omega^q(gl_2)$. Let us set

$$\langle B, B'\rangle=\frac{1}{4\pi}\int_M \mathrm{Tr}(B\wedge B')$$

if $p+q=3$, and $\langle B, B'\rangle=0$ if $p+q\neq 3$. Then

$$\langle B, B'\rangle=\langle B', B\rangle\,. \tag{2.6}$$

Moreover, we have

$$\langle B_1, B_2\wedge B_3\rangle=\langle B_2, B_3\wedge B_1\rangle=\langle B_3, B_1\wedge B_2\rangle\,, \tag{2.7}$$

where $B_j\in\Omega^*(gl_2)$, $j=1, 2, 3$.

LEMMA 2.2. *Let A be a connection on P. Then, for any $B\in\Omega^p(gl_2)$ and for any $B'\in\Omega^{2-p}(gl_2)$, $0\leqq p\leqq 2$, we have*

$$\langle D_AB, B'\rangle=-(-1)^p\langle B, D_AB'\rangle\,.$$

Proof. Integrating the identity

$$d\mathrm{Tr}(B\wedge B')=\mathrm{Tr}(dB\wedge B')+(-1)^p\mathrm{Tr}(B\wedge dB')$$

over M, we first get $\langle dB, B'\rangle=-(-1)^p\langle B, dB'\rangle$. Then, using (2.6) and (2.7), we can prove easily Lemma 2.2.

In terms of the bilinear form, the Chern-Simons action L is written as

$$L(A)=\langle A, dA\rangle+\frac{2}{3}\langle A, A\wedge A\rangle\,.$$

Then, for any $B\in\Omega^1(su_2)$, we get

$$L(A+B)=L(A)+2\langle B, F(A)\rangle+\langle B, D_AB\rangle+\frac{2}{3}\langle B, B\wedge B\rangle\,, \tag{2.8}$$

and hence

$$(dL)_A(B)=2\langle B, F(A)\rangle\,. \tag{2.9}$$

Let us take a Riemannian metric on M and denote by $*$ the Hodge star operator from $\Omega^p(su_2)$ to $\Omega^{3-p}(su_2)$. Then the natural L^2-inner product $(\ ,\ )$ on $\Omega^p(su_2)$ is given by

$$(B, B')=-\langle B, *B'\rangle\,,\qquad B, B'\in\Omega^p(su_2)\,. \tag{2.10}$$

Suppose A is flat. From Lemma 2.2 and (2.10), we see that the formal adjoint operator D_A^* of $D_A^{(p-1)}$ is given by $(-1)^p*D_A*$. Then, by the Hodge theory, $H_A^p(M, su_2)$ is isomorphic with $H_A^{3-p}(M, su_2)$. In particular,

$$d^1(A) = d^2(A). \tag{2.11}$$

Next we shall find another kind of gauge invariants. As was mentioned before, $(\sigma_1, \sigma_2, \sigma_3)$ forms a basis of su_2. Let Ω^p denote the vector space of all usual p-forms on M. We define a linear differential operator $\partial: \Omega^p(su_2) \to \Omega^{p+1}$ by

$$\partial B = d(\omega^1 + \omega^2 + \omega^3),$$

where $B = \omega^1 \cdot \sigma_1 + \omega^2 \cdot \sigma_2 + \omega^3 \cdot \sigma_3$, $\omega^j \in \Omega^p$, $1 \leqq j \leqq 3$. Using ∂, we define a linear subspace of $\Omega^1(su_2)$ by

$$\Omega_A^1(su_2) = \{B \in \Omega^1(su_2) \mid \partial D_A B = 0\}. \tag{2.12}$$

If $A = 0$, then $\Omega_0^1(su_2) = \Omega^1(su_2)$. Let Q'_A denote the quadratic form on $\Omega_A^1(su_2)$ given by $Q'_A(B) = \langle B, D_A B \rangle$, $B \in \Omega_A^1(su_2)$. For the trivial connection $A = 0$, we write simply $Q' = Q'_0$.

LEMMA 2.3. *There are infinite-dimensional linear subspaces $\mathscr{V}$, $\mathscr{V}_+$ and $\mathscr{V}_-$ of $\Omega^1(su_2)$ such that*

1) $\mathscr{V}_\pm \subset \mathscr{V}$;
2) *for any* $A \in \mathscr{A}$, $\mathscr{V} \subset \Omega_A^1(su_2)$;
3) *for any* $A \in \mathscr{A}$, $Q'_A = Q'$ *on* $\mathscr{V}$;
4) Q' *is positive definite on* $\mathscr{V}_+$;
5) Q' *is negative definite on* $\mathscr{V}_-$.

Proof. Let $\sigma = \sigma_1 + \sigma_2 + \sigma_3$, and let $\mathscr{V}$ denote the linear subspace of $\Omega^1(su_2)$ consisting of all elements $B \in \Omega^1(su_2)$ of the form $B = \omega \cdot \sigma$, where $\omega \in \Omega^1$. Then, by a simple calculation, we can prove 2) of Lemma 2.3. Moreover, for any $B \in \mathscr{V}$, we get

$$Q'_A(B) = -\frac{3}{2\pi} \int_M \omega \wedge d\omega,$$

which shows 3) of Lemma 2.3. Now we choose a 1-form ω_+ on M in such a way that

a) the restriction of ω_+ to the open set U of all points x in M with $(\omega_+)_x \neq 0$ is a contact form on U;

b) $\displaystyle\int_M \omega_+ \wedge d\omega_+ < 0.$

(It can be proved that M always admits a global contact form.) Let $\mathscr{V}_+$ denote the linear subspace of $\mathscr{V}$ consisting of all elements $(f\omega_+)\sigma$, where f is an arbitrary smooth function on M with $\mathrm{supp}(f) \subset U$. Then it is easy to see that $\mathscr{V}_+$ satisfies 4). In a similar way, we can find an infinite-dimensional linear subspace $\mathscr{V}_-$ of $\mathscr{V}$ satisfying 5).

Suppose A is flat. For a positive number a, we denote by $\mathscr{R}_{\pm a}(A)$ the set of all elements $B \in \Omega_A^1(su_2)$ with $Q'_A(B) = \pm a$. Moreover, define

(2.13)
$$\mathscr{R}_{\pm a}=\bigcap_{A\in\mathscr{F}}\mathscr{R}_{\pm a}(A)\,.$$

By Lemma 2.3, we can conclude that $\mathscr{R}_{\pm a}$ is not empty and that it is an infinite-dimensional object. Thus the quadratic form Q'_A on $\Omega^1_A(su_2)$ has a huge futile part which does not change as A varies. Moreover, Q'_A has a huge totally singular subspace in $\Omega^1_A(su_2)$.

We now wish to find an essential part which changes as A varies. To do so, we would like to represent Q'_A as a quadratic form on a finite-dimensional vector space. A candidate for such a vector space is either $H^1_A(M, su_2)$ or $H^1(M, su_2)$. But Q'_A cannot descend to these spaces. So we shall find a useful subspace of $H^1(M, su_2)$. Let $Z^1_A(su_2)$ denote the linear subspace of $\Omega^1(su_2)$ consisting of all elements $B\in\Omega^1(su_2)$ such that $dB=0$ and $dA\wedge B=B\wedge dA$. Then, from (2.1), we have

LEMMA 2.4. *If $B\in Z^1_A(su_2)$, then $dD_AB=0$ and hence $\partial D_AB=0$.*

Thus $Z^1_A(su_2)$ is contained in $\Omega^1_A(su_2)$. By Lemma 2.4, we can define a linear map

(2.14)
$$\phi_A:\ Z^1_A(su_2)\to H^2(M, su_2)$$

by $\phi_A(B)=[D_AB]$, $B\in Z^1_A(su_2)$. If $B\in\mathrm{Ker}(\phi_A)$, then $Q'_A(B)=0$ by Lemma 2.2. So the quantity $\mathrm{rank}(\phi_A)$ may be related to the 'rank' of Q'_A.

Now we define

$$V_A(M)=Z^1_A(su_2)/\mathrm{Im}(d_0^{(0)})\cap Z^1_A(su_2)\,.$$

It is clear that $V_A(M)$ is a linear subspace of $H^1(M, su_2)$. By (2.6) and Lemmas 2.2 and 2.4, we can prove easily the following

LEMMA 2.5. *Let A be a flat connection on P. Then there is a unique symmetric bilinear form I_A on $V_A(M)$ such that*

$$I_A([B], [B'])=\langle B, D_AB'\rangle$$

for all $B, B'\in Z^1_A(su_2)$.

By virtue of Lemma 2.5, we can consider a quadratic form Q''_A on $V_A(M)$ defined by $Q''_A(\beta)=I_A(\beta, \beta)$, $\beta\in V_A(M)$. Let $(p(A), q(A))$ be the signature of Q''_A. More precisely, for a suitable basis of $V_A(M)$, Q''_A is represented as follows:

$$Q''_A=x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2\,,$$

where we put $p=p(A)$ and $q=q(A)$. We have $0\leqq p(A)+q(A)\leqq d^1(0)$. Since $p(A)$ and $q(A)$ are bounded, we can set

$$p_+(A)=\mathrm{Max}\{p(Ag)\,|\,g\in\mathscr{G}\}\,,$$

$$p_-(A)=\mathrm{Max}\{q(Ag)\,|\,g\in\mathscr{G}\}\,.$$

Clearly, $p_\pm(A)$ is gauge invariant. Moreover, we set

$$p_\pm(M)=\mathrm{Max}\{p_\pm(A)\,|\,A\in\mathscr{F}\}\,.$$

Then we have $p_{\pm}(M)\leqq d^1(0)$. If M is a homology 3-sphere, then $p_{\pm}(M)=0$. But, in general, $p_{\pm}(M)$ is not trivial. For example, let T^3 denote the three-dimensional torus. Then a simple calculation shows that $p_{\pm}(T^3)$ is non-zero.

§3. Feynman measure

In order to integrate the path integral (1.2), we have to consider 'Feynman measure space' or some other measure space. In the study of path integrals, one often uses a Wiener measure. But our integral (1.2) is not a path integral in the strict sense. A particular idea is needed. However, it is very difficult to clarify the true meaning of the elusive concept 'Feynman measure' (cf. [7]). Some physicists investigate path integrals without knowing an explicit definition of 'Feynman measure'. By a careful examination of their calculations, we can recognize that they assume implicitly many formulas in ordinary integration theory to be valid. On the other hand, recall that our main purpose is to obtain a precise definition of $Z(M; k)$. So we assume that 'Feynman measure' is a measure in the ordinary sense.

In the Chern-Simons theory, an integrand that we are considering may be quantized in a suitable sense. We can then expect that the integrand looks like a simple function. If it is true, there is a simple way to construct an ordinary measure space.

We begin with a general remark. Let S be a set and let $\Delta=\{S_j\}_{j\in N}$ be a family of subsets of S, where N denotes the set of natural numbers. Suppose that Δ gives a disjoint decomposition to S. Then the σ-algebra $\mathfrak{M}'$ generated by Δ consists of all sets T of the form

$$T=\bigcup_{j\in\Lambda} S_j,$$

where Λ is an arbitrary subset of N. We have immediately the following

LEMMA 3.1. *Let $\{a_j\}_{j\in N}$ be a series of non-negative numbers such that $a_j=0$ whenever S_j is empty. Then there is a unique measure μ' on $\mathfrak{M}'$ satisfying $\mu'(S_j)=a_j$ for any $j\in N$.*

Let $(\mathfrak{M}, \mu)$ denote the completion of $(\mathfrak{M}', \mu')$. We shall call $(\mathfrak{M}, \mu)$ the measure space associated with $(\Delta, \{a_j\})$.

Now we consider the moduli space $\mathcal{M}$ of flat connections on P. Let $\alpha, \alpha'\in\mathcal{M}$. We write $\alpha\sim\alpha'$ if there is a piecewise smooth curve A_t, $0\leqq t\leqq 1$, in $\mathscr{A}$ such that

a) A_t is flat for any t with $0\leqq t\leqq 1$;

b) $\alpha=[A_0]$, $\alpha'=[A_1]$.

Then the relation $\sim$ is an equivalence relation in $\mathcal{M}$. Each equivalence class will be called a component of $\mathcal{M}$. Let $\{\mathcal{M}_\lambda\}$ be the set of all components of $\mathcal{M}$. Here we impose the following condition on $\mathcal{M}$:

(C) $\{\mathcal{M}_\lambda\}$ is a finite set.

Let $\{\mathcal{M}_\lambda\}=\{\mathcal{M}_1, \cdots, \mathcal{M}_N\}$, and let $\nu(M)$ denote the minimum value of $d^1(A)$ on $\mathscr{F}$.

Then $\nu(M)$ is a topological invariant. We set $\nu(A)=d^1(A)-\nu(M)$. For non-negative integers h, m, n_+ and n_- with $0\leqq n_\pm\leqq p_\pm(M)$, we denote by $\mathcal{M}_{j,h,m,n_+,n_-}$ the set of all points $[A]$ in $\mathcal{M}_j$ satisfying $h^1(A)=h$, $\nu(A)=m$ and $p_\pm(A)=n_\pm$. Then $\mathcal{M}_j$ is decomposed as

$$\mathcal{M}_j=\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n_+=0}^{p_+(M)}\sum_{n_-=0}^{p_-(M)}\mathcal{M}_{j,h,m,n_+,n_-}\,.$$

Thus $\Delta=\{\mathcal{M}_{j,h,m,n_+,n_-}\}$ determines a disjoint decomposition of $\mathcal{M}$. We set

$$\delta_{j,h,m,n_+,n_-}=\begin{cases}0 & \text{if } \mathcal{M}_{j,h,m,n_+,n_-}=\varnothing\,,\\ 1 & \text{if } \mathcal{M}_{j,h,m,n_+,n_-}\neq\varnothing\,,\end{cases}$$

and define

$$\mu''(\mathcal{M}_{j,h,m,n_+,n_-})=(e/2)^{m/4}\cdot\exp(n_++n_--h)\cdot\delta_{j,h,m,n_+,n_-}\,. \tag{3.1}$$

Then, by Lemma 3.1, μ'' can be extended uniquely to a measure on the σ-algebra generated by Δ. Let $(\mathfrak{M}, \mu)$ denote the measure space associated with (Δ, μ''). Notice that $(\mathfrak{M}, \mu)$ is independent of the orientation of M (see (5.4) and (5.5)). We would expect that $(\mathfrak{M}, \mu)$ might be contained in 'Feynman measure space'.

In addition to the measure space, we need 'Feynman measure' of some kinds of sets. The principle of gauge invariance and the independence of dummy variables will enable us to determine such measures. Of course, we can construct a measure space on each set in such a way that the total measure coincides with our definition.

Let A be a flat connection on P. We reconsider $\Omega_A^1(su_2)$ (see (2.12)) and set

$$\mathcal{V}_A=\Omega_A^1(su_2)/\mathrm{Ker}(D_A)\,. \tag{3.2}$$

There is a unique quadratic form Q_A on $\mathcal{V}_A$ such that

$$Q_A([B])=\langle B, D_AB\rangle\,,\qquad B\in\Omega_A^1(su_2)\,. \tag{3.3}$$

For a non-negative number a, we denote by $\mathcal{S}_{\pm a}(A)$ the subset of $\mathcal{V}_A$ given by

$$\mathcal{S}_{\pm a}(A)=\{\zeta\in\mathcal{V}_A\,|\,Q_A(\zeta)=\pm a\}\,. \tag{3.4}$$

Then $\mathcal{S}_{\pm a}(A)$ is not empty (cf. Lemma 2.3). To define a measure of $\mathcal{S}_{\pm a}(A)$, we use the quantity $p_\pm(A)$. But other contribution must be deliberated. Let $K_A^1(su_2)$ denote the linear subspace of $\Omega^1(su_2)$ consisting of all elements $B\in\Omega^1(su_2)$ such that $dD_AB=0$. Then, from Lemma 2.4, we have

$$Z_A^1(su_2)\subset K_A^1(su_2)\subset\Omega_A^1(su_2)\,.$$

Define a linear map $\psi_A\colon K_A^1(su_2)\to H^2(M, su_2)$ by $\psi_A(B)=[D_AB]$, $B\in K_A^1(su_2)$. This is an extension of the map ϕ_A given by (2.14). Instead of the 'rank' of the quadratic form Q'_A on $\Omega_A^1(su_2)$, we use the quantity

$$r(A)=\mathrm{rank}(\psi_A)\,.$$

From (2.11), we get $r(A)\leqq d^1(0)$. Another contribution comes from the set $\mathcal{R}_{\pm a}$ (see

(2.13)). Now we define a measure $m(\mathscr{S}_{\pm a}(A))$ by

$$m(\mathscr{S}_{\pm a}(A))=(\sqrt{a})^{d^1(A)+1}\cdot\exp(r(A)+p_{\pm}(A)). \tag{3.5}$$

Here the factor $(\sqrt{a})^{d^1(A)+1}$ represents the contribution of the image of $\mathscr{R}_{\pm a}$ in $\mathscr{V}_A$ by the natural projection of $\Omega_A^1(su_2)$ onto $\mathscr{V}_A$.

Next we put

$$\mathscr{W}_A=\Omega^1(su_2)/K_A^1(su_2).$$

If A is the trivial connection, then $\mathscr{W}_A=0$. On the contrary, it can be seen that $\mathscr{W}_A$ is infinite-dimensional for any non-trivial flat connection A. Thus, as is pointed out by Witten, the trivial connection looks very singular. A finite-dimensional vector space which is compared with $\mathscr{W}_A$ is given by

$$K_A^1(su_2)/\mathrm{Ker}(\psi_A)\cong\mathrm{Im}(\psi_A).$$

Set $\mathscr{W}_A^0=\mathscr{W}_A-\{0\}$. Then, for any non-trivial flat connection A, we define $m(\mathscr{W}_A^0)$ by

$$m(\mathscr{W}_A^0)=\frac{1}{\Gamma(\lambda(A))}\cdot\exp(-r(A)), \tag{3.6}$$

where $\Gamma(z)$ is the gamma function and $\lambda(A)$ is given by

$$\lambda(A)=\frac{3}{2}+\frac{1}{2}d^1(A). \tag{3.7}$$

Let $\Omega_A^0(\boldsymbol{H})$ denote the linear subspace of $\Omega^0(\boldsymbol{H})$ consisting of all elements $X\in\Omega^0(\boldsymbol{H})$ such that $A\wedge\mathrm{Tr}(dX)=0$. Clearly, $\mathrm{Ker}(d_0^{(0)})$ is a linear subspace of $\Omega_A^0(\boldsymbol{H})$. So we can consider the following infinite-dimensional vector space:

$$\mathscr{X}_A=\Omega_A^0(\boldsymbol{H})/\mathrm{Ker}(d_0^{(0)}).$$

LEMMA 3.2. *There is a unique bilinear form ξ_A on $\mathscr{W}_A\times\mathscr{X}_A$ such that*

$$\xi_A([C],[X])=\langle C, D_A dX\rangle$$

for any $C\in\Omega^1(su_2)$ and any $X\in\Omega_A^0(\boldsymbol{H})$. Moreover, if an element c of $\mathscr{W}_A$ satisfies $\xi_A(c,x)=0$ for any $x\in\mathscr{X}_A$, then $c=0$.

Proof. It will be sufficient to prove the last assertion, which is restated as follows. If $C\in\Omega^1(su_2)$ satisfies $\langle C, D_A dX\rangle=0$ for any $X\in\Omega_A^0(\boldsymbol{H})$, then $C\in K_A^1(su_2)$. First of all, from Lemma 2.2, we get $\langle dD_A C, X\rangle=0$. Observe that $\Omega^0(su_2)$ is contained in $\Omega_A^0(\boldsymbol{H})$. Then, taking a Riemannian metric on M and using (2.10), we have $(*dD_A C, X)=0$ for any $X\in\Omega^0(su_2)$. Hence we can conclude that C belongs to $K_A^1(su_2)$.

Let A be a non-trivial flat connection on P. For a real number a and $c\in\mathscr{W}_A$, we set

$$\Pi_a(A, c)=\{x\in\mathscr{X}_A \mid \xi_A(c, x)=a\}.$$

From Lemma 3.2, we see that if $c\neq 0$ then $\Pi_a(A, c)$ is a proper affine subspace of $\mathscr{X}_A$. Imagine that $\mathscr{X}_A$ were a finite-dimensional Euclidean space. Then the distance between the origin $0\in\mathscr{X}_A$ and $\Pi_a(A, c)$ is given by $|a|/|\xi_A(c, n)|$, where n is a unit normal vector to $\Pi_a(A, c)$. To define a measure of $\Pi_a(A, c)$, we may use such a quantity. But we replace $|a|/|\xi_A(c, n)|$ by $|a|\exp(-h^1(A))$. A finite-dimensional vector space which is compared with $\mathscr{X}_A$ is given by $H^0(M, \boldsymbol{H})$. Notice that $\dim H^0(M, \boldsymbol{H})=4$. Then, for any non-trivial flat connection A, we define $m(\Pi_a(A, c))$ by

$$m(\Pi_a(A, c))=\frac{\sqrt{2}}{3}\cdot|a|^3\cdot\exp(-3h^1(A)) \qquad (c\neq 0). \tag{3.8}$$

§4. Evaluation of the Feynman path integral

We are now in a position to evaluate the integral (1.2). As we can not obtain a reasonable measure space on $\mathscr{C}$, we have to reform the definition of $Z(M; k)$. Here we use an idea in quantum field theory.

Let $\pi: \mathscr{A}\to\mathscr{C}$ be the canonical projection. Suppose there is a gauge-fixing submanifold $\mathscr{N}$ of $\mathscr{A}$. This means that $\mathscr{N}$ determines a cross section of π. Set $\mathscr{N}_0=\mathscr{N}\cap\mathscr{F}$. For the moment, we assume that $\mathscr{N}$ satisfies the following condition: There is a linear subspace $\mathscr{U}$ of $\Omega^1(su_2)$ such that

1) $\Omega^1(su_2)=\mathrm{Ker}(D_A)\oplus\mathscr{U}$ for any $A\in\mathscr{N}_0$;

2) to each $A'\in\mathscr{N}$, there correspond a unique point $A\in\mathscr{N}_0$ and a unique vector $B\in\mathscr{U}$ in such a way that

$$A'=A+B.$$

Regarding this as a change of coordinates, we know that $\mathscr{D}A'$ is proportional to $\mathscr{D}A\mathscr{D}B$:

$$\mathscr{D}A'=J^*\mathscr{D}A\mathscr{D}B.$$

Here J^* corresponds to the so-called Faddeev-Popov determinant and is represented by 'ghost fields' (see [9]). Moreover, J^* depends only on A. Then the definition of $Z(M; k)$ is divided into two steps:

$$H(A)=J^*(A)\int_{\mathscr{U}}\exp(ikL(A+B))\mathscr{D}B, \tag{4.1}$$

$$Z(M; k)=\int_{\mathscr{N}_0}H(A)\mathscr{D}A. \tag{4.2}$$

If we identify $\mathscr{N}_0$ with $\mathscr{M}$, then (4.2) may be replaced by

$$Z(M;k)=\int_{\mathscr{M}} H(A)\mathscr{D}[A]\,. \tag{4.3}$$

There may be no canonical gauge-fixing submanifold $\mathscr{N}$ satisfying the assumption. However, the formulas (4.1) and (4.3) are useful for our purpose. So we want to define $Z(M;k)$ by using (4.1) and (4.3). But, in general, $\mathscr{U}$ may change as A varies. Then the problem is to replace $\mathscr{U}$ by a reasonable family $\{\mathscr{U}_A\}$ of vector spaces so that $H(A)$ is gauge invariant. To find such a family, we take a Riemannian metric on M. Then the tangent space $T_A(\mathscr{A})$ has the Hodge decomposition:

$$T_A(\mathscr{A})=\mathrm{Ker}(D_A)\oplus\mathrm{Im}(D_A^*)\,.$$

We denote by $\mathscr{L}(A)$ the subset of $\mathscr{A}$ consisting of all points of the form $A+B\in\mathscr{A}$, $B\in\mathrm{Im}(D_A^*)$, and by $\mathscr{C}(A)$ the image of $\mathscr{L}(A)$ in $\mathscr{C}$ by the projection π. Then it can be seen that $\mathscr{C}(Ag)=\mathscr{C}(A)$ for any $g\in\mathscr{G}$. Though we can put $\mathscr{U}_A=\mathrm{Im}(D_A^*)$, there are some inevitable faults:

a) $\mathscr{U}_A$ depends on the choice of a metric on M.

b) The restriction of π to $\mathscr{L}(A)$ is not necessarily injective.

c) Even if two flat connections A_1 and A_2 are not gauge equivalent, the sets $\mathscr{C}(A_1)$ and $\mathscr{C}(A_2)$ have an overlap.

By these facts, we can recognize that $\{\mathrm{Im}(D_A^*)\}$ cannot be a desired family. To remove the fault a), we may consider the following vector space:

$$\mathscr{E}_A=\Omega^1(su_2)/\mathrm{Ker}(D_A)\,,$$

which is isomorphic with $\mathrm{Im}(D_A^*)$. But there is still a question. Since A is flat, it follows from (2.8) that $L(A+B)$ is expanded in powers of B as follows:

$$L(A+B)=L(A)+\langle B, D_AB\rangle+\frac{2}{3}\langle B, B\wedge B\rangle\,.$$

Then it can be seen that $L(A+B)$ can not descend to $\mathscr{E}_A$ without using a Riemannian metric on M. We must cut off the cubic term. But, if we do so, we can not obtain a correct value for the integral. To regulate such a difficulty and the faults b) and c), we have to replace $\mathscr{E}_A$ by some other vector space.

Now we make use of the vector space $\mathscr{V}_A$ defined by (3.2). We think of $\mathscr{V}_A$ as being a reasonable infinitesimal gauge fixing at A. Moreover, instead of $L(A+B)$, we use the function $L(A)+Q_A$ on $\mathscr{V}_A$, where Q_A is the quadratic form on $\mathscr{V}_A$ given by (3.3). So we define $H(A)$ by

$$H(A)=J(A)\int_{\mathscr{V}_A}\exp(ik(L(A)+Q_A(\beta)))\mathscr{D}\beta\,. \tag{4.4}$$

Here $J(A)$ consists of a factor corresponding to the Faddeev-Popov determinant and a correction factor with respect to the faults b) and c). In order to determine $J(A)$ explicitly, we use the vector spaces $\mathscr{W}_A$ and $\mathscr{X}_A$ and the bilinear form ξ_A (see §3). Set

$\mathscr{W}_A^0=\mathscr{W}_A-\{0\}$. Then, taking account of the Faddeev-Popov construction, we define $J(A)$ by setting

$$J(A)=\int_{\mathscr{W}_A^0}\left(\int_{\mathscr{X}_A}\exp(i\xi_A(c,x))\mathscr{D}x\right)\mathscr{D}c\,. \tag{4.5}$$

Now we define $Z(M;k)$ by the formulas (4.3), (4.4) and (4.5). But it should be remarked that $H(A)$ can not be properly defined for the trivial connection. Since $\mathscr{W}_0^0$ is empty and $\xi_0=0$, we may put $H(0)=0$ or $H(0)=\infty$. At any rate, the trivial connection is a singular point of $H(A)$. So, for the moment, we assume that A is a non-trivial flat connection. However, we shall see later that the trivial connection is a removable singularity.

First of all, we integrate $\exp(ikQ_A(\beta))$ over $\mathscr{V}_A$. Let a be a positive number and let $\mathscr{S}_{\pm a}(A)$ be the subset of $\mathscr{V}_A$ given by (3.4). Consider the cone spanned by $\mathscr{S}_{\pm a}(A)$:

$$\mathscr{C}_\pm(A)=\{r\zeta\in\mathscr{V}_A\,|\,r>0,\,\zeta\in\mathscr{S}_{\pm a}(A)\}\,.$$

Clearly, $\mathscr{C}_\pm(A)$ does not depend on $a>0$. Then $\mathscr{C}_\pm(A)$ and $\mathscr{S}_0(A)$ determine a disjoint decomposition of $\mathscr{V}_A$. Since $m(\mathscr{S}_0(A))=0$ by (3.5), the integral over $\mathscr{S}_0(A)$ vanishes. So we have only to evaluate the integral

$$H_\pm(A)=\int_{\mathscr{C}_\pm(A)}\exp(ikQ_A(\beta))\mathscr{D}\beta\,.$$

Consider the change of coordinates $\beta=r\zeta$ on $\mathscr{C}_\pm(A)$, where $r>0$ and $\zeta\in\mathscr{S}_{\pm a}(A)$. Our task is then to find its Jacobian.

Let us here digress to discuss a finite-dimensional version. For simplicity, we consider the three-dimensional Euclidean space $\boldsymbol{R}^3$ and an oriented surface S in $\boldsymbol{R}^3$. Suppose S is given by

$$\boldsymbol{x}(u,v)=(x(u,v),\,y(u,v),\,z(u,v))\,.$$

If S is in a general position with respect to the origin, a generic point (X,Y,Z) in a region of $\boldsymbol{R}^3$ can be represented as

$$X=rx(u,v)\,,\quad Y=ry(u,v)\,,\quad Z=rz(u,v)\,.$$

Then we have

$$dX\wedge dY\wedge dZ=r^2(\boldsymbol{x},\boldsymbol{n})dr\wedge dS\,,$$

where $\boldsymbol{n}$ is a unit normal vector field to S and dS is the volume form of S.

We would expect that a similar formula might hold for our case. But the matter is not so simple. First, the factor r^2 should be replaced by r^n, where n is an integer. As a finite-dimensional vector space which is compared with $\mathscr{V}_A$ is probably given by $H_A^1(M,su_2)$, we may put $n=d^1(A)-1$. But this definition does not yield a consistent measure space on $\mathscr{V}_A$. We can see that n should satisfy $n\geqq 2$. So let us set $n=d^1(A)+2$. To examine the factor $(\boldsymbol{x},\boldsymbol{n})$, we take a Riemannian metric on M and regard $\mathscr{V}_A$ as a linear subspace of $\mathrm{Im}(D_A^*)$. $\mathscr{S}_{\pm a}(A)$ is then given by the equation

$\langle B, D_A B\rangle = \pm a$, $B \in \mathscr{V}_A$. From (2.10), we can see that every tangent vector V to $\mathscr{S}_{\pm a}(A)$ at B satisfies $(V, *D_A B) = 0$. Then

$$N = *D_A B / \| *D_A B \|$$

is a unit normal vector to $\mathscr{S}_{\pm a}(A)$ at B, where $\| \cdot \|$ denotes the L^2-norm on $\Omega^1(su_2)$. Hence,

$$(B, N) = -\frac{1}{\| *D_A B \|} \langle B, D_A B\rangle .$$

But the factor $\| *D_A B \|^{-1}$ is irrelevant.

By these consideration, we obtain

$$\mathscr{D}\beta = r_a^{d^1(A)+2} \cdot | Q_A(\zeta_a) | \mathscr{D}r_a \mathscr{D}\zeta_a , \tag{4.6}$$

where $(r_a, \zeta_a) \in (0, \infty) \times \mathscr{S}_{\pm a}(A)$. Notice that $\mathscr{C}_{\pm}(A)$ is identified with $(0, \infty) \times \mathscr{S}_{\pm a}(A)$. We think of $\mathscr{D}r_a$ as being the Gaussian measure $\exp(-ar^2)dr$, where dr denotes the usual Lebesgue measure. Consider the following transformation:

$$r_b = \sqrt{\frac{a}{b}} \cdot r_a , \quad \zeta_b = \sqrt{\frac{b}{a}} \cdot \zeta_a \qquad (a, b > 0) .$$

Then $\sqrt{a}\, \mathscr{D}r_a$ is left invariant by the transformation. Taking account of the definition (3.5), $\mathscr{D}\zeta_a$ must change as follows:

$$\mathscr{D}\zeta_b = \left(\sqrt{\frac{b}{a}}\right)^{d^1(A)+1} \cdot \mathscr{D}\zeta_a . \tag{4.7}$$

We can construct a suitable measure space on $\mathscr{S}_{\pm a}(A)$ satisfying (3.5) and (4.7). Then the result (4.6) does not depend on the choice of a and determines a measure space on $\mathscr{V}_A$. It may be unnecessary to discuss this matter in detail.

By (4.6), we can rewrite $H_{\pm}(A)$ as

$$H_{\pm}(A) = \int_{\mathscr{S}_{\pm a}(A)} \left(a \int_0^\infty r^{d^1(A)+2} e^{\pm iakr^2} \mathscr{D}r \right) \mathscr{D}\zeta . \tag{4.8}$$

Here we make use of the following formula of Laplace transform:

$$\int_0^\infty e^{-\varepsilon t} t^{\lambda - 1} e^{iut} dt = \frac{\Gamma(\lambda)}{(u^2 + \varepsilon^2)^{\lambda/2}} \cdot e^{i\lambda\theta} , \tag{4.9}$$

where $\varepsilon > 0$, $\lambda > 0$ and θ is the principal value of $\tan^{-1}(u/\varepsilon)$. Putting $t = r^2$, $\varepsilon = a$ and $u = \pm ak$, we get

$$\int_0^\infty r^{d^1(A)+2} e^{\pm iakr^2} \mathscr{D}r = \frac{\Gamma(\lambda(A))}{2a^{\lambda(A)}(k^2+1)^{\lambda(A)/2}} \cdot e^{\pm i\lambda(A)\theta(k)} ,$$

where $\lambda(A)$ is the quantity given by (3.7) and $\theta(k)$ is the principal value of $\tan^{-1}(k)$. From the definition (3.5) and (4.8), we can deduce

$$H_{\pm}(A)=\frac{1}{2}(k^2+1)^{-\lambda(A)/2}\cdot\Gamma(\lambda(A))\exp(r(A)+\tau_{\pm}(A)),$$

where we put $\tau_{\pm}(A)=p_{\pm}(A)\pm i\lambda(A)\theta(k)$. To simplify the notation, we set

$$U(A)=\frac{1}{2}(\exp\tau_{+}(A)+\exp\tau_{-}(A)). \tag{4.10}$$

Then we finally have

$$\int_{\mathscr{V}_A}\exp(ikQ_A(\beta))\mathscr{D}\beta=(k^2+1)^{-\lambda(A)/2}\cdot\Gamma(\lambda(A))U(A)e^{r(A)}. \tag{4.11}$$

Next we shall evaluate $J(A)$. First, we integrate $\exp(i\xi_A(c,x))$ over $\mathscr{X}_A$. To do so, we reconsider the affine subspace $\Pi_{\pm a}(A,c)$ of $\mathscr{X}_A$, where $c\in\mathscr{W}_A^0$ and a is a positive number. For simplicity, we write $\Pi_{\pm a}=\Pi_{\pm a}(A,c)$. As before, we set

$$\mathscr{H}_{\pm}=\{ty\in\mathscr{X}_A\mid t>0,\ y\in\Pi_{\pm a}\}.$$

Then $\mathscr{H}_{\pm}$ and Π_0 determine a disjoint decomposition of $\mathscr{X}_A$. Since $m(\Pi_0)=0$ by (3.8), the problem is reduced to the evaluation of the integral

$$J_{\pm}(A)=\int_{\mathscr{H}_{\pm}}\exp(i\xi_A(c,x))\mathscr{D}x.$$

Let us consider the change of coordinates $x=ty$ on $\mathscr{H}_{\pm}$, where $t>0$ and $y\in\Pi_{\pm a}$. Notice that $\dim H^0(M,\boldsymbol{H})=4$. Then, as in the preceding case, we obtain

$$\mathscr{D}x=t_a^3\,|\,\xi_A(c,y_a)\,|\,\mathscr{D}t_a\mathscr{D}y_a, \tag{4.12}$$

where $(t_a,y_a)\in(0,\infty)\times\Pi_{\pm a}$. Here we think of $\mathscr{D}t_a$ as being the Laplace measure $\exp(-at)dt$. As before, we can arrange $\mathscr{D}y_a$ so that (4.12) is independent of a. Then $\mathscr{D}x$ determines a measure space on $\mathscr{X}_A$.

Now we get

$$J_{\pm}(A)=\int_{\Pi_{\pm a}}\left(a\int_0^{\infty}t^3e^{\pm iat}\mathscr{D}t\right)\mathscr{D}y. \tag{4.13}$$

Putting $\varepsilon=a$, $\lambda=4$ and $u=\pm a$ in (4.9), we obtain

$$\int_0^{\infty}t^3e^{\pm iat}\mathscr{D}t=-\frac{3}{2a^4}. \tag{4.14}$$

By (4.5), (4.13), (4.14) and the definitions (3.6) and (3.8), we find

$$J(A)=-\frac{\sqrt{2}}{\Gamma(\lambda(A))}\cdot\exp(-r(A)-3h^1(A)).$$

Then, by (4.11), we can conclude that

$$H(A)=-\sqrt{2}(k^2+1)^{-\lambda(A)/2}\cdot U(A)\exp(ikL(A)-3h^1(A))\,. \tag{4.15}$$

So far, we assume that A is a non-trivial flat connection. But the result (4.15) can be well applied to the trivial connection. It should be also remarked that $H(A)$ is really gauge invariant.

Finally, we shall integrate $H(A)$ over $\mathcal{M}$. Using the formula (2.9), we can prove easily the following

LEMMA 4.1. *Let A and A' be flat connections on P. If there is a piecewise smooth curve A_t, $0\leqq t\leqq 1$, in $\mathcal{A}$ satisfying $A_0=A$, $A_1=A'$ and $A_t\in\mathcal{F}$ for all t, then $L(A)=L(A')$.*

We write $A\sim A'$ if A and A' satisfy the condition in Lemma 4.1. This is clearly an equivalence relation and induces the equivalence relation in $\mathcal{M}$ which we considered in §3. Here we use freely the notation and the terminology in §3. From Lemma 4.1, we have

LEMMA 4.2. *The function* $\exp(ikL(A))$ *on $\mathcal{M}$ remains constant on each component of $\mathcal{M}$.*

As before, we impose the following condition on $\mathcal{M}$:

(C) $\{\mathcal{M}_\lambda\}$ is a finite set.

Then we obtain the measure space $(\mathfrak{M},\mu)$. Let $\{\mathcal{M}_1,\cdots,\mathcal{M}_N\}$ be the set of all components of $\mathcal{M}$. By Lemma 4.2, we can fix an element A_j of $\mathcal{M}_j$ so that

$$\exp(ikL(A))=\exp(ikL(A_j))$$

for any $A\in\mathcal{M}_j$. Let $\nu(M)$ be the minimum value of $d^1(A)$ on $\mathcal{F}$. Set

$$\lambda(M)=\frac{3}{2}+\frac{1}{2}\nu(M) \tag{4.16}$$

and $\nu(A)=d^1(A)-\nu(M)$. Then we have $\lambda(A)=\lambda(M)+(\nu(A)/2)$. Let $F_\pm(A)$ denote the gauge invariant function on $\mathcal{F}$ given by

$$F_\pm(A)=(k^2+1)^{-\nu(A)/4}\cdot\exp\{p_\pm(A)-3h^1(A)\pm\frac{1}{2}i\nu(A)\theta(k)\}\,.$$

Since $F_\pm(A)$ is regarded as a function on $\mathcal{M}$ and is a measurable function with respect to the measure space $(\mathfrak{M},\mu)$, we can consider the integral

$$\Lambda_j^\pm(k)=\int_{\mathcal{M}_j}F_\pm(A)\mathcal{D}[A]\,.$$

By the definition (3.1), we find without difficulty

$$\Lambda_j^\pm(k)=\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n_+=0}^{p_+}\sum_{n_-=0}^{p_-}\sigma(k)^m\cdot e^{\pm im\theta/2}\cdot a_{j,h,m,n_+,n_-}^\pm\,,$$

where we put $p_\pm=p_\pm(M)$, $\theta=\theta(k)$,

$$\sigma(k)=\left(\frac{e}{2k^2+2}\right)^{1/4}$$

and

$$a^{\pm}_{j,h,m,n_+,n_-}=\exp(n_+ + n_- + n_{\pm} - 4h)\cdot\delta_{j,h,m,n_+,n_-}\,.$$

Recall that k is assumed to be a positive integer. Then the infinite series converges absolutely, so the integral really makes sense. If we put

$$a^{\pm}_{j,m}=\sum_{h=0}^{\infty}\sum_{n_+=0}^{p_+}\sum_{n_-=0}^{p_-}a^{\pm}_{j,h,m,n_+,n_-}\,,$$

then $\Lambda^{\pm}_j(k)$ is written as

$$\Lambda^{\pm}_j(k)=\sum_{m=0}^{\infty}\sigma(k)^m a^{\pm}_{j,m}\cdot e^{\pm im\theta(k)/2}\,.$$

To simplify the notation, we set

$$\chi_j(k)=\frac{-1}{\sqrt{2}}(\Lambda^+_j(k)\exp(i\lambda(M)\theta(k))+\Lambda^-_j(k)\exp(-i\lambda(M)\theta(k)))\,. \tag{4.17}$$

Now, by the result (4.15) and (4.10), we can conclude finally that

$$Z(M;k)=(k^2+1)^{-\lambda(M)/2}\cdot\sum_{j=1}^{N}\exp(ikL(A_j))\cdot\chi_j(k)\,. \tag{4.18}$$

Since $\Lambda^{\pm}_j(k)$ is somewhat complicated, we consider the large k limit of $Z(M;k)$. Clearly, we have

$$\lim_{k\to\infty}\theta(k)=\pi/2 \quad\text{and}\quad \lim_{k\to\infty}\sigma(k)=0\,.$$

So we set

$$\chi_{j,\infty}=\frac{-1}{\sqrt{2}}(a^+_{j,0}\cdot\exp(i\pi\lambda(M)/2)+a^-_{j,0}\cdot\exp(-i\pi\lambda(M)/2))$$

and define

$$Z(M;k)_{\infty}=k^{-\lambda(M)}\cdot\sum_{j=1}^{N}\exp(ikL(A))\cdot\chi_{j,\infty}\,. \tag{4.19}$$

If k is sufficiently large, then $Z(M;k)$ may be replaced by $Z(M;k)_{\infty}$. But $Z(M;k)_{\infty}$ is defined for any positive integer k.

Let us now consider the case where M is a homology 3-sphere satisfying $d^1(A)=0$ for any flat connection A (cf. [8]). Suppose further that M satisfies the condition (C). This implies that $\mathscr{M}$ is a finite set. We have $p_{\pm}(M)=0$, $\nu(M)=0$ and $\lambda(M)=3/2$. In this case, $\Lambda^{\pm}_j(k)$ is a finite series and is given by

$$A_j^{\pm}(k) = a_{j,0}^{\pm} = \sum_h e^{-4h} \cdot \delta_{j,h,0,0,0} \,.$$

Hence, $\chi_j(k) = -\sqrt{2}\, a_{j,0}^{+} \cdot \cos(3\theta(k)/2)$. From (4.18), we obtain

$$Z(M;k) = -\sqrt{2}\,(k^2+1)^{-3/4} \cdot \cos(3\theta(k)/2) \sum_j \exp(ikL(A_j)) \cdot a_{j,0}^{+}$$

and

$$Z(M;k)_\infty = k^{-3/2} \cdot \sum_j \exp(ikL(A_j)) \cdot a_{j,0}^{+} \,.$$

In particular, we have

$$Z(S^3;k) = -\sqrt{2}\,(k^2+1)^{-3/4} \cdot \cos(3\theta(k)/2)$$

and hence

$$Z(S^3;k)_\infty = k^{-3/2} \,.$$

§5. Invariants for three-manifolds

In the course of the study, we have met with some invariants for three-manifolds. Here we shall show some properties of main quantities.

Under the finiteness condition (C), $Z(M;k)$ has been obtained as

$$Z(M;k) = (k^2+1)^{-\lambda(M)/2} \sum_{j=1}^{N} \exp(ikL(A_j)) \cdot \chi_j(k) \,,$$

where $\lambda(M)$ and $\chi_j(k)$ are given by (4.16) and (4.17), respectively. Let M^* denote the manifold M with the opposite orientation. We first prove the following

THEOREM 5.1. *Let M be a compact, connected, oriented three-manifold satisfying the condition* (C). *Then*:

1) $Z(M^*;k) = \overline{Z(M;k)}$.
2) $Z(M;k)$ *is a topological invariant.*

Proof. We denote the quantities associated with M^* by adding the symbol $*$. The fundamental formula is

$$\langle B, B' \rangle^* = -\langle B, B' \rangle \,, \qquad B, B' \in \Omega^*(gl_2) \,.$$

Then we obtain the following formulas:

$$L^*(A) = -L(A) \,, \qquad A \in \mathscr{A} \,. \tag{5.1}$$

$$p_{\pm}^*(A) = p_{\mp}(A) \,, \qquad A \in \mathscr{F} \,. \tag{5.2}$$

$$p_{\pm}(M^*) = p_{\mp}(M) \,. \tag{5.3}$$

(5.4) $$\mathscr{M}^*_{j,h,m,n_+,n_-}=\mathscr{M}_{j,h,m,n_-,n_+}\,.$$

(5.5) $$\delta^*_{j,h,m,n_+,n_-}=\delta_{j,h,m,n_-,n_+}\,.$$

On the other hand, we have $d^1(A)^*=d^1(A)$ and hence $\lambda(M^*)=\lambda(M)$. Now, using (5.2)$\sim$(5.5), we can verify $\Lambda_j^{\pm}(k)^*=\overline{\Lambda_j^{\mp}(k)}$. Hence we have $\chi_j(k)^*=\overline{\chi_j(k)}$. Then the assertion 1) follows easily from (5.1).

We can see that $Z(M;k)$ is an invariant under orientation preserving diffeomorphisms. But it is known that every topological three-manifold has a differentiable structure unique up to diffeomorphism (see for example [10] p. 544). Thus 2) follows from this fact.

In a similar way, we can see that Theorem 5.1 holds also for the quantity $Z(M;k)_\infty$ given by (4.19).

PROPOSITION 5.2. *Let M be as in Theorem* 5.1. *If M admits an orientation reversing diffeomorphism, then both* $Z(M;k)$ *and* $Z(M;k)_\infty$ *are real numbers.*

Proof. Every orientation reversing diffeomorphism of M determines an orientation preserving diffeomorphism from M to M^*. Then the assertion follows immediately from Theorem 5.1.

We now define real numbers $I(M;k)$ and $R(M;k)$ by setting

$$Z(M;k)=R(M;k)+iI(M;k)\,.$$

Then, by Theorem 5.1, $I(M;k)$ and $R(M;k)$ are topological invariants for M satisfying

$$I(M^*;k)=-I(M;k)\,,\qquad R(M^*;k)=R(M;k)\,.$$

Thus $R(M;k)$ defines a proper topological invariant.

Although $p_\pm(M)$ is a topological invariant, we are interested in the quantity

$$\mathrm{sgn}(M)=p_+(M)-p_-(M)\,,$$

which is defined without any additional condition on M. From (5.3), we get

$$\mathrm{sgn}(M^*)=-\mathrm{sgn}(M)\,.$$

Moreover, we can prove easily the following

PROPOSITION 5.3. *Let M be a compact, connected, oriented three-manifold. If M admits an orientation reversing diffeomorphism, then* $\mathrm{sgn}(M)=0$.

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